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# MINIMAL GENERALIZED COMPUTABLE ENUMERATIONS AND HIGH DEGREES

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**Abstract:** We establish that the set of minimal generalized computable enumerations of every infinite family computable with respect to a high oracle is effectively infinite. We find some sufficient condition for enumerations of the infinite families computable with respect to high oracles under which there exist minimal generalized computable enumerations that are irreducible to the enumerations.

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This article deals with the minimal enumerations of families which admit uniform enumerations with respect to high oracles. The choice of this class of oracles is motivated by active research into computable enumerations of families of arithmetic sets [1–5] introduced by Goncharov and Sorbi [1]. An enumeration $\alpha$ of a family $\mathscr{A}$ is called $\Sigma^0_n$-*computable*, where $n \geq 1$, whenever

$$G_\alpha = \{\langle x, y\rangle : y \in \alpha x\} \in \Sigma^0_n.$$

A broader approach to generalizing the concept of computable enumeration is the concept of enumeration computable with an oracle. Following [1], say that an enumeration $\alpha$ is $X$-*computable* whenever $G_\alpha$ is computably enumerable with oracle $X$ (also $X$-computably enumerable). Therefore, for each $n$ the class of all $\Sigma^0_{n+2}$-computable enumerations coincides with the class of enumerations computable with respect to the high oracle $\varnothing^{(n+1)}$.

As regards, the background on enumeration theory, see the book by Ershov [6] and his article [7]; for computability theory see the book of Soare [8]. Say that an enumeration $\alpha$ *is reducible to* an enumeration $\beta$ whenever $\alpha = \beta \circ f$ for some computable function $f$. Two enumerations $\alpha$ and $\beta$ are called *equivalent*, written $\alpha \equiv \beta$, whenever $\alpha \leq \beta$ and $\beta \leq \alpha$. Call an enumeration $\alpha$ of a family $\mathscr{A}$ *minimal* if $\alpha \leq \beta$ for every enumeration $\beta \leq \alpha$ of the same family. Given an oracle $X$, denote by $W^X_e$ the $X$-computably enumerable set with the Gödel number $e$. Given a set $Y$ and a number $x$, put $Y^{\langle x\rangle} = \{y : \langle x, y\rangle \in Y\}$. Given a set $X$ and a number $e$, define the $X$-computable enumeration $\alpha^X_e$ by putting $\alpha^X_e x = \big(W^X_e\big)^{\langle x\rangle}$ for all $x$. Given an $X$-computable family $\mathscr{A}$, define the index set

$$\mathrm{Min}^X(\mathscr{A}) = \big\{e : \alpha^X_e \text{ is a minimal enumeration of } \mathscr{A}\big\}.$$

The question of existence and cardinality of minimal $\Sigma^0_{n+2}$-computable enumerations of an infinite family of arithmetic sets is completely settled by Badaev and Goncharov [2]. They established that each infinite $\Sigma^0_{n+2}$-computable family has infinitely many inequivalent $\Sigma^0_{n+2}$-computable minimal enumerations and asked the questions of effective infinity of the set of minimal $\Sigma^0_{n+2}$-computable enumerations and a description of $\Sigma^0_{n+2}$-computable enumerations whose lower cones avoid certain minimal $\Sigma^0_{n+2}$-computable operations. Aiming at the problem of effective infinity of the set of minimal computable enumerations of infinite families with respect to a high oracle, we can prove the lemma.

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